

NOTE

A Counterexample to the Countable Version of a Conjecture of Ulam

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A conjecture of Ulam [2] states that, if G and G' are two graphs with point sets $\{v_i\}_{i=1,\dots,p}$ and $\{v'_i\}_{i=1,\dots,p}$, $p \geq 3$, such that, for each i , $G - v_i$ is isomorphic to $G' - v'_i$, then G is isomorphic to G' . Harary [1] asks whether this is true for infinite graphs; it is the purpose of this note to give a counterexample.

We construct the graphs G and G' as follows: The points of G are the points (m, n) of the Euclidean plane with integral coordinates $m \geq 1$, $n \geq 0$, and an additional point s above the plane. An edge is drawn from (p, q) to $(p + 1, q)$ unless $p = aq$ for some integer a ; an edge is drawn from (p, q) to s if p is odd. G' is the subgraph of G obtained by removing $(1, 0)$. The points of G' are denoted with primes, as s' , $(4, 5)'$, etc.

Clearly G and G' are not isomorphic. In fact, any isomorphism would have to map s to s' , and the maximal infinite tree in $G - s$ to the maximal infinite tree in $G' - s'$. This would then require that $(1, 0)$ be mapped onto $(1, 1)'$, which is a contradiction, as the two points have different degrees.

Rather than actually constructing numberings of the point of G and G' with the property that corresponding subgraphs are isomorphic, we remark that it is sufficient to count the number of each isomorphism type of subgraph and to notice that there is an equal number of each type in G and G' ; a correspondence will then exist.

The removal of s from G yields the only subgraph with no points of infinite degree, and this clearly corresponds to the removal of s' from G' . The removal of any point with a non-zero second coordinate from G will yield a subgraph isomorphic to G , for the finite tree in the plane in which the point appears will split into two smaller such trees, each of which already appears countably often. Similarly, when $(p, 0)$ is removed, the finite end, i.e., the points $(q, 0)$, $q < p$, will be "absorbed" by the rest of

the graph, to yield a subgraph isomorphic to G if p is even and to G' if p is odd. Thus there are countably many subgraphs isomorphic to G , and countably many isomorphic to G' . The same argument will produce countably many subgraphs of G' isomorphic to each of G and G' . A one-to-one correspondence therefore exists between the points of the non-isomorphic graphs G and G' , such that the removal of any pair of corresponding points yields isomorphic subgraphs.

REFERENCES

1. F. HARARY, On the Reconstruction of a Graph from a Collection of Subgraphs, *Theory of Graphs and Its Applications* (M. Fiedler, ed.), Academic Press, New York, 1965, pp. 47–52.
2. S. M. ULAM, *A Collection of Mathematical Problems*, Wiley, New York, 1960, p. 29.